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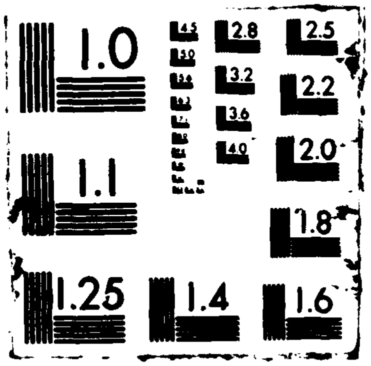
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EXPLICIT FORMULAE FOR THE DISTRIBUTIONS OF STOPPING VARIABLES UNDER WALD'S TRUNCATED SPRT FOR A POISSON PROCESS

by

S. Zacks

GWU/IMSE/Serial T-516/87
6 May 1987

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
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Key Words: Stopping variables, Wald SPRT, Poisson processes, MTBF

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1. Introduction

The computation of probabilities associated with sequential stopping variables can often be easily performed by recursive techniques (e.g. Armitage et al (1969), Aroian and Robinson (1969), Samuel-Cahn (1974), Pocock (1977), Zacks (1974, 1980, 1981, 1987)). Explicit formulae for probabilities of boundary crossings are generally difficult to obtain. Wiener process and other asymptotic approximations to the distributions of stopping times are discussed by Siegmund (1985). The present paper shows that in the case of sequential procedures based on Poisson processes, explicit expression can be derived. These derivations are facilitated by the nondecreasing nature of the sample paths, the memory-less property of the Poisson process, and the possibility to solve certain recursive relations associated with the probability distributions of the stopping variables. For the purpose of showing the potential of this technique, we derive explicit expressions for the cumulative distribution functions (CDF) of stopping variables associated with the acceptance and with the rejection boundaries of a Wald's SPRT, which is frequency truncated. Such stopping boundaries are applied in reliability life testing for systems having

exponentially distributed life length (see MIL-STD 781C (1977)). Methods of determining confidence intervals for the MTBF, after such sequential life testing, were given by Siegmund (1978), Bryant and Schmee (1979), and Zacks (1987). Siegmund approximated the distributions of stopping times, while Zacks determined them recursively. The results of the present paper eliminate the need for recursive computations in the Poisson case.

In Section 2 we present the stopping boundaries, the observed process and some general definitions. In Section 3 we derive explicitly certain boundary crossing probabilities. Section 4 is devoted to an actual derivation of the distributions of the stopping variables, for a particular case under consideration. A numerical example is provided in Section 5.

2. The Failure Process and Stopping Variables

Let $\{X(t): 0 < t < \infty\}$ be a Poisson process with intensity λ, $0 < \lambda < \infty$; i.e., $E_\lambda\{X(t)\} = \lambda t$, $0 < t < \infty$. We consider a truncated Wald SPRT, which is specified by the boundary lines

$$b_R(t) = \begin{cases} k_1 + bt, & 0 < t \leq t_{k_s - k_1} \\ k_s, & t_{k_s - k_1} \leq t \leq t_{k_s + k_0} \end{cases} \tag{2.1}$$

and

$$b_A(t) = -k_0 + bt, \quad t_{k_0} \leq t \leq t_{k_0 + k_s}, \tag{2.2}$$

where k_1, k_0, k_s are integers, $k_s > k_1$, and $t_i = i/b$ $(i = 1, 2, \ldots)$. The boundary $b_A(t)$ is called the acceptance boundary, and $b_R(t)$ is called the rejection boundary. Notice that $b_A(t)$ and $b_R(t)$ intersect at $(t_{k_0+k_s}, k_s)$.

We define the stopping times (variables)

$$\tau_R = \inf\left\{t:\ X(t) \geq b_R(t),\ 0 < t < t_{k_0+k_s}\right\}, \tag{2.3}$$

$$\tau_A = \inf\left\{t:\ X(t) = b_A(t),\ t_{k_0} \leq t \leq t_{k_0+k_s}\right\}, \tag{2.4}$$

where $\{X(t)\colon t > 0\}$ is a nondecreasing jump process, with jumps of one unit. The process is stopped at the first instant τ at which it crosses either the lower boundary, $b_A(t)$, or the upper boundary $b_R(t)$. Obviously, $\tau = \min\,(\tau_A, \tau_R)$.

For the derivation of the distributions of the stopping times τ_R, τ_A and τ one has to distinguish between three cases,

$$\text{Case I:} \quad k_1 < k_s \le k_1 + k_0;$$
$$\text{Case II:} \quad k_1 + k_0 < k_s \le 2k_1 + k_0;$$
$$\text{Case III:} \quad 2k_1 + k_0 < k_s.$$

Since the method of derivation of the distributions is similar in all the three cases, we focus attention in the present paper only on Case I.

The sample paths of the Poisson process $\{X(t)\colon 0 < t\}$ assume only nonnegative integer values. Thus, a sample path can cross the lower boundary $b_A(t)$ only at the discrete times t_i, $k_0 \le i \le k_s + k_0 - 1$. Let

$$f^{(\lambda)}_{A,k_0}(j) = P_\lambda\{\tau_A = t_{k_0+j}\}$$

be the probability distribution function of the discrete stopping variable τ_A, over t_{k_0+j}. We introduce the notation

$$I\{A\} = \begin{cases} 1, & \text{if } A \text{ is true} \\ 0, & \text{otherwise;} \end{cases}$$

and $p(j;\xi)$ and $Pos(j;\xi)$ are to be, respectively, the PDF and CDF of the Poisson distribution with mean ξ.

Let $F^{(\lambda)}_A(t)$ denote the corresponding CDF of τ_A. The stopping time τ_R on the other hand, has an absolutely continuous distribution. Define the probability function

$$\text{(2.5)} \qquad g_\lambda(j;\, t) = P_\lambda\{X(t) = j,\ \tau \ge t\}, \quad j = 0, 1, \ldots$$

Let [a] be the largest integer smaller than or equal to a, and define

$$M(t) = \begin{cases} [k_1 + bt], & \text{if } k_1 + bt \text{ is not an integer} \\ k_1 + bt - 1, & \text{otherwise.} \end{cases} \tag{2.6}$$

Similarly, let $a^+ = \max\{a, 0\}$ and let

$$m(t) = \begin{cases} [-k_0 + bt], & \text{if } t \geq t_{k_0} \\ [-k_0 + bt]^+ - 1, & \text{if } t < t_{k_0} \end{cases} \tag{2.7}$$

Then

$$P_\lambda\{\tau > t\} = \sum_{j=m(t)+1}^{M(t)} g_\lambda(j; t). \tag{2.8}$$

Let $F_R^{(\lambda)}$ designate the CDF of τ_R, then

$$F_R^{(\lambda)}(t) = 1 - F_A^{(\lambda)}(t) - P_\lambda\{\tau > t\}, \quad 0 < t. \tag{2.9}$$

Furthermore, from the Markovian property of the Poisson process, for each $t_{i-1} < t \leq t_i$

$$F_R^{(\lambda)}(t) = F_R^{(\lambda)}\left(t_{i-1}\right) + \sum_{j=m(t_{i-1})+1}^{M(t_{i-1})} g_\lambda(j; t_{i-1}) \left[1 - \mathrm{Pos}\left(M(t_{i-1}) + 1 - j; \lambda(t - t_{i-1})\right)\right] \tag{2.10}$$

This formula holds for i = 1 too, if we define $g_\lambda(j; t_0) = I\left\{j = 0\right\}$. Notice that $F_R^{(\lambda)}(t)$ is a differentiable function of t, for $t_{i-1} < t < t_i$ $(i = 1, 2, \ldots, k_S + k_0)$. Thus, $F_R^{(\lambda)}(t)$ is absolutely continuous. In Section 4 we develop explicit formulas for $F_A^{(\lambda)}(t)$ and $F_R^{(\lambda)}(t)$, appropriate for Case I.

3. Linear Boundaries Crossing Probabilities

Let h_λ (i, s) designate the probability that a Poisson process, with intensity λ, assumes the value i at time s, and does not cross the linear boundary $x(u) = bu$ for $0 < u \leq s$, i. e.,

$$(3.1) \qquad h_\lambda(i; s) = P_\lambda\left\{\sup(X(u) - bu < 0; 0 < u < s), X(s) = i\right\}.$$

Let $\mathfrak{U}_s^-$ be the region below x(u) over (0, s], and $\mathfrak{U}_s^+$ the region above it (including the boundary), i. e.,

$$(3.2) \qquad \mathfrak{U}_s^- = \{(u, x): \ 0 \leq x < bu, \ 0 < u \leq s\}$$

and

$$(3.3) \qquad \mathfrak{U}_s^+ = \{(u, x): \ bu \leq x, \quad 0 < u \leq s\}.$$

Conditioning on the level at which the last entrance of a sample path to $\mathfrak{U}_s^-$ occurs, we obtain the recursive equation

$$(3.4) \qquad h_\lambda(i; s) = \begin{cases} p(0; \lambda s), & \text{if } i = 0 \\ 0, & \text{if } i > 0, s \leq t_i \\ p(i; \lambda s) - \sum_{\ell=1}^{i} p(\ell; \lambda t_\ell)\, h_\lambda(i - \ell; s - t_\ell), & \text{if } i > 0, s > t_i \end{cases}$$

Let $\mu = \lambda/b$ and let $H_\lambda(i; s) = \sum_{j=0}^{i} h_\lambda(j; s)$. Thus, for each j=0, 1, 2,...

and $i = j + 1, j + 2, \ldots$ we obtain the recursive relation

$$(3.5) \qquad H_\lambda(j; t_i) = \begin{cases} p(0; i\mu), & j = 0 \\ \mathrm{Pos}(j; i\mu) - \sum_{\ell=1}^{j} p(\ell; \mu\ell)\, H_\lambda(j - \ell; t_{i-\ell}), & j \geq 1 \end{cases}$$

An explicit solution of (3.5) can be obtained by using the theory of formal power-series (see P. Henrici (1974); p. 17).

For $\theta \in (-1, 1)$ and $\delta = 0, 1, 2, \ldots$ define the power-series:

$$
\begin{aligned}
H^*_{\lambda,\delta}(\theta) &= \sum_{j=0}^{\infty} H_\lambda(j;\, t_{j+\delta})\, \theta^j, \\
F^*_{\lambda,\delta}(\theta) &= \sum_{j=0}^{\infty} \mathrm{Pos}(j;\, \mu(j+\delta))\, \theta^j, \qquad (3.6)\\
P^*_{\lambda,\delta}(\theta) &= \sum_{j=0}^{\infty} p(j;\, \mu(j+\delta))\, \theta^j,
\end{aligned}
$$

where $h_\lambda(0; 0) = 1$ and $p(0, 0) = 1$. The power-series $P^*_{\lambda,0}(\theta)$ has an inverse, given by $Q^*_\lambda(\theta) = \sum_{n=0}^{\infty} q_n^{(\lambda)}\, \theta^n$, where

$$
(3.7) \qquad q_n^{(\lambda)} = \begin{cases} 1, & n = 0 \\ -\,p(1;\mu), & n = 1 \\ (-1)^n D_n(\mu), & n \geq 2 \end{cases}
$$

where $D_n(\mu)$ is the determinant of the n x n matrix

$$
(3.8) \quad A_n(\mu) = \begin{bmatrix} p(1;\mu) & p(2;2\mu) & \ldots & p(n;n\mu) \\ 1 & p(1;\mu) & \ldots & p(n-1;(n-1)\mu) \\ 0 & 1 & \ldots & p(n-2;(n-2)\mu) \\ 0 & 0 & \ldots & p(n-3;(n-3)\mu) \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & p(1;\mu) \end{bmatrix}
$$

From (3.5) we obtain the equation

$$
(3.9) \qquad H^*_{\lambda,\delta}(\theta) = F^*_{\lambda,\delta}(\theta) - H^*_{\lambda,\delta}(\theta)\,(P^*_{\lambda,0}(\theta) - 1),
$$

or, equivalently,

$$
(3.10) \qquad H^*_{\lambda,\delta}(\theta)\, P^*_{\lambda,0}(\theta) = F^*_{\lambda,\delta}(\theta).
$$

Hence,

$$H^*_{\lambda,\delta}(\theta) = F^*_{\lambda,\delta}(\theta)\, Q^*_\lambda(\theta) \tag{3.11}$$

and, for each $i = 0, 1, \ldots$

$$H_\lambda(i; t_{i+\delta}) = \sum_{n=0}^{i} q_n^{(\lambda)} \operatorname{Pos}(i - n; (i - n + \delta)\mu). \tag{3.12}$$

In a similar fashion we obtain that for every $j = 0, \ldots, i - 1$ $(i = 1, 2, \ldots)$

$$h_\lambda(j; t_i) = \sum_{n=0}^{j} q_n^{(\lambda)}\, p(j - n; (i - n)\mu). \tag{3.13}$$

Consider a linear boundary $x_\ell(u) = -\ell + bu$, for an integer value ℓ. Let $f^{(\lambda)}_{A,\ell}(i)$ be the probability that the process crosses this boundary for the first time at $(t_{\ell+i}, i)$, $i = 0, 1, 2, \ldots$ These boundary crossing probabilities satisfy the following recursive equation for $i = 0, 1, \ldots$

$$f^{(\lambda)}_{A,\ell}(i) = p(i; (\ell + i)\mu) - I\{i > 1\} \sum_{j=0}^{i-1} f^{(\lambda)}_{A,\ell}(j)\, p(i - j; (i - j)\mu). \tag{3.14}$$

Define the power-series, for $\theta \in (-1, 1)$,

$$f^*_{\lambda,\ell}(\theta) = \sum_{i=0}^{\infty} f^{(\lambda)}_{A,\ell}(i)\, \theta^i. \tag{3.15}$$

The recursive equation (3.14) yields the equation

$$f^*_{\lambda,\ell}(\theta) = P^*_{\lambda,\ell}(\theta) \left[P^*_{\lambda,0}(\theta)\right]^{-1}, \tag{3.16}$$

where $\left[P^*_{\lambda,0}(\theta)\right]^{-1} \equiv Q^*_\lambda(\theta)$. Hence, we obtain by multiplying the power-series,

$$f^{(\lambda)}_{A,\ell}(i) = \sum_{n=0}^{i} q_n^{(\lambda)}\, p(i - n; (\ell + i - n)\mu), \quad \text{for } i = 0, 1, \ldots \tag{3.17}$$

4. The Distributions of Stopping Times in Case I

The continuation region (between $b_A(t)$ and $b_R(t)$) is partitioned into four regions:

$$R_1 = \{(t, x): \quad 0 < t \le t_{k_s - k_1}, \quad 0 \le x < b_R(t)\};$$

$$R_2 = \{(t, x): \quad t_{k_s - k_1} < t \le t_{k_0}, \quad 0 \le x < b_R(t)\};$$

$$R_3 = \{(t, x): \quad k_0 < t \le t_{k_0 + k_1}, \quad b_A(t) < x < b_R(t)\}; \text{ and}$$

$$R_4 = \{(t, x): \quad t_{k_0 + k_1} < t \le t_{k_s + k_0}, \; b_A(t) < x < b_R(t)\}.$$

4.1 Distributions over R_1

Over R_1, $\tau = \tau_R$, while $P_\lambda\{\tau = \tau_A\} = 0$. From the Markovian properties we obtain, for $i = 1, \ldots, k_s - k_1$, $j = 0, 1, \ldots, k_1 + i - 1$,

$$\text{(4.1)} \quad g_\lambda(j; t_i) = p(j; \mu i) - I\{j \ge k_1 + 1\} \sum_{\ell=1}^{j-k_1} p(k_1 + \ell; \ell\mu)\, h_\lambda(j - k_1 - \ell; t_i - t_\ell).$$

Hence, for $1 \le i \le k_s - k_1$,

$$\text{(4.2)} \quad P_\lambda\{\tau_k > t_i\} = \mathrm{Pos}(k_1 + i - 1; i\mu) - I\{i \ge 2\} \sum_{j=k_1+1}^{k_1+i-1} \sum_{\ell=1}^{j-k_1} p(k_1 + \ell; \ell\mu)\, h_\lambda(j - k_1 - \ell; t_i - t_\ell).$$

Changing the order of summation we obtain

$$\text{(4.3)} \quad P_\lambda\{\tau_R > t_i\} = \mathrm{Pos}(k_1 + i - 1; i\mu) - I\{i \ge 2\} \sum_{\ell=1}^{i-1} p(k_1 + \ell; \ell\mu)\, H_\lambda(i - 1 - \ell; t_{i-\ell}).$$

Thus, for every $i = 1,\ldots, k_S - k_1$, we obtain from (4.3) and (3.12),

$$(4.4)\quad F_A^{(\lambda)}(t_i) = 1 - \mathrm{Pos}(k_1 + i - 1; i\mu)$$

$$+ I\{i \geq 2\}$$

$$\cdot \sum_{\ell=1}^{i-1} p(k_1 + \ell; \ell\mu) \sum_{n=0}^{i-1-\ell} q_n^{(\lambda)} \ \mathrm{Pos}(i - 1 - \ell - n; (i - \ell - n)\mu).$$

4.2 <u>The Distributions over R_2</u>

Over R_2 we obtain similar results to those over R_1, with only slight changes. Here, the CDF of τ_A is

$$(4.5)\qquad F_A^{\lambda}(t_i) = \begin{cases} 0, & i = k_S - k_1 + 1,\ldots, k_0 - 1 \\ f_{A,k_0}^{(\lambda)}(0), & i = k_0. \end{cases}$$

The CDF of τ_R is given by

$$(4.6)\ F_R^{(\lambda)}(t_i) = 1 - \mathrm{Pos}(k_S - 1; i\mu)$$

$$+ I\{k_S > k_1 + 1\} \sum_{\ell=1}^{k_S-k_1-1} p(k_1 + \ell; \ell\mu)\, H_\lambda(k_S - k_1 - \ell - 1; t_{i-\ell})$$

for $i = k_S - k_1 + 1,\ldots, k_0$. According to (3.12),

$$(4.7)\qquad H_\lambda(k_S - k_1 - \ell - 1; t_{i-\ell})$$

$$= \sum_{n=0}^{k_S-k_1-\ell-1} q_n^{(\lambda)} \ \mathrm{Pos}(k_S - k_1 - \ell - n - 1; (i - \ell - n)\mu).$$

4.3 <u>The Distributions over R_3</u>

The probability distribution (PDF) of τ_A over R_3 is given by

$$(4.8)\qquad P_\lambda\{\tau_A = k_0 + i\} = f_{A,k_0}^{(\lambda)}(i),\ i = 1,\ldots, k_1,$$

where $f^{(\lambda)}_{A,k_0}(i)$ is given by (3.17).

The probabilities $g_\lambda\left(j;\, t_{k_0+i}\right)$, for $i \le i \le k_1$, are given by

(4.9) $$g_\lambda\left(j;\, t_{k_0} + i\right) = I\{1 \le j \le k_1\}$$

$$\cdot \left(p\left[j;\, \mu(k_0 + i)\right] - \sum_{\ell=0}^{i-1} f^{(\lambda)}_{A,k_0}(\ell)\, p\left[j - \ell;\, (j - \ell)\mu\right] \right)$$

$$+ I\{k_1 + 1 \le j \le k_g - 1\} \cdot p\left[j;\, \mu(k_0 + i)\right]$$

$$- I\{k_1 + 1 \le j \le k_g - 1\}$$

$$\cdot \sum_{\ell=0}^{i-1} f^{(\lambda)}_{A,k_0}(\ell)\, p\left[j - \ell;\, \mu(i - \ell)\right]$$

$$- I\{k_1 + 1 \le j \le k_g - 1\}$$

$$\cdot \sum_{m=1}^{j-k_1} p\left[k_1 + m;\, m\mu\right] h_\lambda\left(j - k_1 - m;\, t_{k_0+i-m}\right)$$

where, according to (3.13), for $k_1 + 1 \le j \le k_g - 1$,

(4.10) $$h_\lambda\left(j - k_1 - m;\, t_{k_0+i-m}\right)$$

$$= \sum_{n=0}^{j-k_1-m} q^{(\lambda)}_n\, p\left[j - k_1 - m - n;\, \mu(k_0 + i - m - n)\right].$$

From (4.9) we obtain, for $i \leq i \leq k_1$,

(4.11) $$F_R^{(\lambda)}(t_i) = 1 - F_A^{(\lambda)}(t_i) - P_\lambda\{\tau > t_i\}$$

$$= 1 - F_A^{(\lambda)}(t_i) - \mathrm{Pos}(k_s - 1; \mu(k_0 + i)) + \mathrm{Pos}(i; \mu(k_0 + i))$$

$$+ \sum_{\ell=0}^{i-1} f_{A,k_0}^{(\lambda)}(\ell)\, [\mathrm{Pos}(k_s - 1 - \ell; \mu(i - \ell)) - \mathrm{Pos}(i - \ell; \mu(i - \ell))]$$

$$+ I\{k_s > k_1 + 1\}$$

$$\cdot \sum_{\ell=1}^{k_s-k_1-1} p(k_1 + \ell; \ell\mu)\, H_\lambda\{k_s - k_1 - \ell - 1; t_{k_0+i-\ell}\}.$$

4.4 The Distributions over R_4

We assume that $k_s > k_1 + 1$. If $k_s = k_1 + 1$ then $P_\lambda\{\tau \leq t_{k_0+k_1}\} = 1$.

Applying the Markovian property of the Poisson process, we obtain for

$1 \leq i \leq k_s - k_1 - 1$,

(4.12) $$P_\lambda\{\tau_A = t_{k_0+k_1} + i\} = f_{A,k_0}^{(\lambda)}(k_1 + i)$$

$$= \sum_{\ell=1}^{i} g_\lambda(k_1 + \ell; t_{k_0+k_1})$$

$$\cdot \left[p(i - \ell; \mu i) - I\{i > \ell\} \sum_{m=0}^{i-\ell-1} f_{A,\ell}^{(\lambda)}(m)\, p[i - \ell - m; \mu(i - \ell - m)]\right]$$

where $g_\lambda(k_1 + \ell; t_{k_0+k_1})$ is given by (4.9).

In a similar manner we establish that, for every $1 \leq i \leq k_s - k_1 - 2$,

$$(4.13)\quad g_\lambda\left(k_1 + j; t_{k_0+k_1+i}\right) = I\left\{i + 1 \leq j \leq k_s - k_1 - 1\right\} \sum_{\ell=1}^{j} g_\lambda\left(k_1 + \ell; t_{k_0+k_1}\right)$$

$$\cdot \left[p(j - \ell; \mu i) - I\left\{\ell < i\right\} \sum_{m=0}^{i-\ell-1} f_{A,\ell}^{(\lambda)}(m)\, p\left(j - \ell - m; \mu(i - \ell - m)\right)\right].$$

Hence,

$$(4.14)\quad P_\lambda\left\{\tau > t_{k_0+k_1+i}\right\} = I\left\{i < k_s - k_1 - 1\right\} \sum_{j=i+1}^{k_s-k_1-1} g_\lambda\left(k_1 + j; t_{k_0+k_1+i}\right)$$

$$= I\left\{i < k_s - k_1 - 1\right\}$$

$$\cdot \left\{ \sum_{\ell=1}^{j} g_\lambda\left(k_1 + \ell; t_{k_0+k_1}\right) \left[Pos\left(k_s - k_1 - \ell - 1; i\mu\right) - Pos(i - \ell; i\mu)\right] \right.$$

$$+ \sum_{\ell=i+1}^{k_s-k_1-1} g_\lambda\left(k_1 + \ell; t_{k_0+k_1}\right) Pos\left(k_s - k_1 - \ell - 1; i\mu\right)$$

$$- I\left\{i > 1\right\} \sum_{m=0}^{i-2} \sum_{\ell=1}^{i-1-m} g_\lambda\left(k_1 + \ell; t_{k_0+k_1}\right) \cdot f_{A,\ell}^{(\lambda)}(m)$$

$$\left. \cdot \left[Pos(k_s - k_1 - \ell - m - 1; \mu(i - \ell - m)) - Pos(i - \ell - m; \mu(i - \ell - m))\right]\right\}.$$

5. Numerical Example

It is a simple matter to compute the CDF of the stopping times (variables) τ_R, τ_A and τ according to the formula given in Section 4. In Table 5.1 we present these distributions for Case I, with $k_0 = 15$, $k_1 = 15$, $k_s = 28$ and $b = 20$. We present $F_R^{(\lambda)}(t)$, $F_A^{(\lambda)}(t)$ and $F^{(\lambda)}(t)$ at the values of $t_i = i/b$ $(i = 1, 2, \ldots, k_s + k_0 - 1)$ for Poisson processes with $\lambda = 10$ and $\lambda = 20$. Notice that the values of λ are specified for a convenient time unit, which is traditionally take to be the MTBF under the alternative hypothesis, and a specified number of systems on test (see Zacks (1987)). The values of Table 5.1 were computed on a PC, using a TURBO-PASCAL program, which is available upon request.

Table 5.1: Cumulative Distributions of Stopping Times
For Truncated Wald SPRT ($k_0=k_1=15$, $k_S=28$, B=20)

t	$\lambda = 10$			$\lambda = 20$		
	$F_R(t)$	$F_A(t)$	$F(t)$	$F_R(t)$	$F_A(t)$	$F(t)$
0.05	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.10	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.15	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.20	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.25	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.30	0.000000	0.000000	0.000000	0.000002	0.000000	0.000002
0.35	0.000000	0.000000	0.000000	0.000005	0.000000	0.000005
0.40	0.000000	0.000000	0.000000	0.000013	0.000000	0.000013
0.45	0.000000	0.000000	0.000000	0.000028	0.000000	0.000028
0.50	0.000000	0.000000	0.000000	0.000056	0.000000	0.000056
0.55	0.000000	0.000000	0.000000	0.000100	0.000000	0.000100
0.60	0.000000	0.000000	0.000000	0.000165	0.000000	0.000165
0.65	0.000000	0.000000	0.000000	0.000258	0.000000	0.000258
0.70	0.000000	0.000000	0.000000	0.000661	0.000000	0.000661
0.75	0.000000	0.000553	0.000553	0.001728	0.000000	0.001729
0.80	0.000000	0.003069	0.003069	0.004111	0.000002	0.004113
0.85	0.000000	0.009555	0.009555	0.008836	0.000007	0.008844
0.90	0.000000	0.022050	0.022050	0.017319	0.000020	0.017339
0.95	0.000001	0.042105	0.042106	0.031269	0.000044	0.031312
1.00	0.000002	0.070480	0.070482	0.052481	0.000085	0.052566
1.05	0.000005	0.107088	0.107094	0.082541	0.000149	0.082690
1.10	0.000013	0.151118	0.151131	0.122503	0.000243	0.122747
1.15	0.000027	0.201242	0.201269	0.172631	0.000373	0.173004
1.20	0.000056	0.255844	0.255900	0.232258	0.000545	0.232803
1.25	0.000109	0.313230	0.313340	0.299814	0.000764	0.300578
1.30	0.000204	0.371778	0.371983	0.372996	0.001035	0.374031
1.35	0.000367	0.430045	0.430412	0.449057	0.001362	0.450420
1.40	0.000635	0.486825	0.487460	0.525136	0.001749	0.526885
1.45	0.001061	0.541171	0.542232	0.598567	0.002198	0.600766
1.50	0.001716	0.592389	0.594105	0.667131	0.002712	0.669843
1.55	0.002692	0.640016	0.642707	0.729195	0.003291	0.732486
1.60	0.004105	0.683788	0.687893	0.783771	0.003936	0.787707
1.65	0.015958	0.713544	0.729502	0.830488	0.004514	0.835003
1.70	0.029219	0.737448	0.766667	0.869395	0.005117	0.874512
1.75	0.042745	0.758349	0.801093	0.901008	0.005788	0.906796
1.80	0.056490	0.777129	0.833618	0.926085	0.006545	0.932630
1.85	0.070690	0.794002	0.864692	0.945507	0.007393	0.952899
1.90	0.085617	0.809013	0.894631	0.960181	0.008329	0.968510
1.95	0.101469	0.822211	0.923680	0.970978	0.009350	0.980328
2.00	0.118164	0.833688	0.951852	0.978670	0.010451	0.989121
2.05	0.134752	0.843573	0.978325	0.983895	0.011627	0.995522
2.10	0.147980	0.852020	1.000000	0.987127	0.012873	1.000000

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